A M I O P A S Q U A L E, N D O & D M A T H E W S

Problem Solving Tactics

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The books in this series are selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solutions. Typically, the problems have occurred in either national or international contests at the secondary school level.

They are intended to be sufficiently detailed at an elementary level for the mathematically inclined or interested to understand but, at the same time, be interesting and sometimes challenging to the undergraduate and the more advanced mathematician. It is believed that these mathematics competition problems are a positive influence on the learning and enrichment of mathematics.
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About this book

What is this book about?

Each year the Australian Mathematical Olympiad Committee (AMOC) runs two training schools. These are designed to extend and challenge the mathematical skills of the 25 secondary school students who are invited to attend. Particular emphasis is given to honing the skill of problem solving.

This book is based on past and present lectures given at the two annual AMOC training schools. As such it is suitable for

- anyone who wishes to qualify for an Olympiad training school in mathematics, either in Australia or overseas
- anyone who has attended an Olympiad training school in mathematics and who would like to be better prepared should they qualify again for an invitation
- interested students, teachers and parents, as it will give an idea of the sorts of mathematics considered there
- any mathematically able students, hobbyists or problem solvers, whether local or abroad, who would find this publication enriching.

What is in this book?

The authors have gone to considerable care to showcase many of the tricks and problem-solving tactics they consider to be important for Olympiad mathematics and problem solving in general. Apart from the first chapter the topics are grouped into the four broad traditional Olympiad divisions of number theory, geometry, algebra and combinatorics.

Most of the sections within each main chapter highlight a particular idea important for problem solving, thus providing over 150 such ideas in total. Each idea is illustrated with one or two problems along with solutions. For extra practice, most chapters begin with a list of problems. Although they are not necessarily in order of difficulty, we have tried to arrange them so that the first few problems tend to be easier than the later ones.

Mathematics can be quite hard to read and digest and so the style has purposely been kept rather informal and conversational. The book often gives the impression that it is conversing with the reader.

How do you use this book?

Most chapters do not depend much on other chapters. Therefore, apart from the first chapter, most can be studied almost independently of each other. However, where dependencies arise there are cross references.

It is the opinion of the authors and many others involved in AMOC training schools that the chief way to improve one’s problem-solving ability is to go through the struggle of trying to solve problems oneself. So we recommend that:

The focus of the user of this book should not be on reading solutions but on trying to solve problems.
That is why solutions are not provided to the problems at the beginning of each chapter. It is also why we recommend that a problem be tried thoroughly with the showcased idea of the section in mind, before the solution is studied.

We recognise that some problems are relatively easy exercises while others are of the difficulty of the International Mathematical Olympiad—the pinnacle of problem-solving mathematics for high school students the world over. So the reader definitely should not expect to be able to solve all of the problems straight away.

Acknowledgments

Some problems are the inventions of staff members at AMOC training schools. However, many of the problems have come from contests such as the Australian Mathematical Olympiad (AMO), national mathematical Olympiads of some other countries, the Asian Pacific Mathematics Olympiad (APMO), the International Mathematical Olympiad (IMO) and problems shortlisted for the IMO. Since many of these problems have appeared in multiple contests, in many cases it has been hard to identify their true origin and so we would simply like to acknowledge all of the above sources.

Although this book has three listed authors, the ideas it contains are the product of many AMOC staff interacting with each other and with students over many years.

We also express our appreciation to Ross Atkins, Andrew Elvey Price, Ivan Guo, Konrad Pilch, Chaitanya Rao, Sally Tsang, Graham White and Sampson Wong, who assisted in proofreading for mathematical content and accuracy, and provided other feedback.

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Norman Do represented Australia at the 1997 International Mathematical Olympiad and was Deputy Leader for the Australian team on four occasions. He obtained a PhD in mathematics from The University of Melbourne in 2010. He has worked at McGill University, Quebec, and is now a lecturer at Monash University, where he researches problems that combine geometry, topology, combinatorics, and mathematical physics. He is currently the Chair of the Australian Mathematical Olympiad Committee’s Senior Problems Committee, which sets national mathematics Olympiad papers and proposes problems for international competitions.

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Proofs are the essence of mathematics. They are nothing more than logical arguments which present the solution to a mathematical problem beyond all possible doubt. There is no set format for a proof, no particular way in which it must be presented on a page. Some people may try to tell you that proofs must appear in two columns, with statements on the left and explanations on the right—but this is complete nonsense! As long as you have sufficiently clear assumptions, a logical argument that is fully explained, and the correct conclusion, then you have a proof.\footnote{There is a rigorous notion of what constitutes a mathematical proof, but for our purposes and for the purposes of most modern mathematics, our informal explanation is sufficient.}

So how do you know whether or not you have a rigorous proof to a mathematical problem? Well, this is a difficult question to answer. Being able to write correct proofs without leaving out important details is a skill which can only be learned through experience—that is, by reading and writing them yourself. However, the following little test might help you on your way. Imagine that you have an inquisitive younger sibling who is looking over your shoulder as you write your proof and constantly interrogating you. ‘What does this mean?’ she might ask, or ‘Why are you doing that?’ or ‘How does that make sense?’ If you claim that something is obvious, she will ask why and if you are tempted to be vague, she just won’t understand. Your goal, of course, is to make sure that she understands and to answer all of her questions before she has even asked them. In order to accomplish this, you should provide a clear explanation for every single fact you write down that is not blatantly obvious.

Apart from the logical aspect of your proof, there are a few other points to keep in mind. Proofs should always be clear, concise, and most definitely legible. Keep in mind that a nice proof to a problem is usually shorter than a messy one. This is often achieved by using good notation and providing the right definitions. A common problem is for people to provide extraneous material, which does not contribute to the main argument, or to write long essays, which are not enlightening at all. One way to avoid these pitfalls is to break a problem into smaller chunks, which can be individually proved and then reassembled to provide the complete proof. In general, a good proof should take the reader on a pleasant mathematical journey ending at the desired result.

### 1.0 Problems

1. Prove that if $a + b$ is an irrational number, then at least one of $a$ or $b$ is irrational.
2. Show that at any party, there are always at least two people with exactly the same number of friends at the party.

3. The equal temperament tuning\(^2\) of musical instruments is based on the fact that \(2^{\frac{19}{12}}\) is very close to 3.

Show that there can be no perfect tuning\(^3\) by proving that if \(2^x = 3\), then \(x\) must be irrational.

4. If \(m\) and \(n\) are positive integers, prove that \(\sqrt[3]{n}\) is either a positive integer or irrational.

5. Prove that there are infinitely many prime numbers of the form \(6n + 5\), where \(n\) is a positive integer.

6. For every positive integer \(n\), prove that
\[
\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{(n-1) \times n} = \frac{n-1}{n}.
\]

7. Prove that \(n^2 < 2^n\) for every integer \(n \geq 5\).

8. For every positive integer \(n\), prove that
\[
1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

9. For every positive integer \(n\), prove that
\[
1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},
\]
and go on to conclude that
\[
1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.
\]

10. Recall that the Fibonacci sequence is defined recursively by \(F_1 = 1\), \(F_2 = 1\) and \(F_{n+1} = F_n + F_{n-1}\), for \(n \geq 2\).

Prove the following identity for Fibonacci numbers: for all \(n \geq 1\),
\[
F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.
\]

11. Prove that every positive integer can be uniquely expressed as a sum of different numbers, where each number is of the form \(2^n\) for some non-negative integer \(n\).

12. Suppose \(x\) is a real number such that \(x + \frac{1}{2}\) is an integer.

Prove that \(x^n + \frac{1}{2^n}\) is also an integer for any positive integer \(n\).

13. Any finite collection of lines in the plane divides the plane up into regions.

Prove that it is possible to colour each of these regions either black or white in such a way that no two regions which share a common edge have the same colour.

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\(^2\)This tuning has perfect octaves and almost perfect fifths. Octaves are tuned so that the ratio of the frequency of the pitch of the higher note to that of the note an octave lower is 2 : 1. Perfect fifths are tuned so that the ratio of the frequency of the pitch of the higher note to the note a fifth below is 3 : 2. Standard equal temperament tuning says that rising by an almost perfect fifth 12 times should be the same as rising by a perfect octave seven times. Thus \((\frac{3}{2})^{\frac{1}{12}} \approx 2^{\frac{1}{12}}\).

\(^3\)By perfect tuning, we mean that all octaves and fifths are perfect.
14. Show that if there are five points in a square with side length 1 metre, then there exist two of them which are less than 75 centimetres apart.

15. Four points are given inside a square with side length 8 metres.
   (a) Prove that two of them are less than $\sqrt{65}$ metres apart.
   (b) Can you prove, beyond a shadow of a doubt, that two of them are less than 8 metres apart?

16. (a) Prove that if $x$ and $y$ can each be written as the sum of the squares of two integers, then so can $xy$.
   (b) Prove that if $x$ and $y$ are both of the form $a^2 + 2b^2$ $(a, b \in \mathbb{Z})$, then so is $xy$.
   (c) Let $k$ be a fixed integer. Prove that if $x$ and $y$ are both of the form $a^2 + kb^2$ $(a, b \in \mathbb{Z})$, then so is $xy$.

17. Consider the non-empty subsets of $\{1, 2, \ldots, n\}$. For each of these subsets, consider the reciprocal of the product of its elements.
   Determine the sum of all of these numbers.

18. A finite set of chords is drawn in a circle such that each of them passes through the midpoint of another chord.
   Prove that all of the chords must be diameters.

19. Show that if we take $n + 1$ numbers from the set $\{1, 2, 3, \ldots, 2n\}$, there must exist two which have no common factor greater than 1.
   Does this remain true if we take $n$ numbers?

20. A circular island is divided into states by a number of chords of the circle. Consider a tour that starts and ends in the same state without passing through the intersection of any two borders.
   Prove that the tour must involve an even number of border crossings.

21. Suppose you are given a balance scale and a collection of weights whose masses are $1, 3, 3^2, 3^3, \ldots$.
   (a) Prove that using these masses you can determine the weight of any object whose mass is a positive integer.
   (b) Prove that apart from interchanging the contents of the left and right pans of the scale, the configuration of masses on the pans that correctly determines the weight is unique.
   (c) Prove that the weights $1, 3, 3^2, 3^3, \ldots$ are the only integral weights that uniquely determine the weight of every integral mass.

22. A group of people played in a tennis tournament where each person played exactly one match against every other person.
   Prove that it is always possible to put the players in a line so that the first player beat the second, the second player beat the third, all the way down to the last player.

23. A polygon is divided into triangles by diagonals whose endpoints are the vertices of the polygon in such a way that no two of the diagonals intersect inside the polygon.
   Prove that it is possible to colour the vertices of the polygon with three colours so that the three vertices of each triangle have different colours.
24. The **Tower of Hanoi** is a mathematical puzzle consisting of three rods and \(n\) discs of distinct sizes, which can slide onto any of the three rods. The puzzle starts with the discs neatly stacked in order of size on one rod, the smallest at the top, as shown in the diagram below. The aim is to transfer the entire stack to another rod by moving a disc from the top of one stack to the top of another stack in such a way that no disc is placed on top of a smaller disc.

Prove that the task can be accomplished in \(2^n - 1\) moves but not in fewer moves.

25. Thirty coins lie on a table, with 17 of them showing heads. Your task, should you choose to accept it, is to separate the coins into two piles, not necessarily of the same size, each of which has the same number of heads showing. Unfortunately, you happen to be blindfolded and cannot feel the difference between the two sides of a coin.

How can you perform the task?

26. Each of the numbers 1, 2, 3, \ldots, \(n^2\) is written in one of the squares of an \(n \times n\) chessboard. Show that there exist two squares which share a vertex or an edge whose entries differ by at least \(n + 1\).

27. Each square of an \(8 \times 8\) chessboard has a real number written in it in such a way that each number is equal to the geometric mean of all the numbers a knight’s move away from it.

Is it true that all of the numbers must be equal?

28. There are 1000 positive numbers written at different points on the circumference of a circle. If the numbers \(x, y, z\) appear in a row in that order, then it is known that \(xz = y^2\).

Prove that all of the numbers are equal.

29. There are \(m\) horizontal lines and \(n\) vertical lines drawn in the plane. Each point of intersection between a pair of lines is coloured in one of 100 colours. Find values of \(m\) and \(n\) such that, no matter how the colouring is performed, there always exists a rectangle whose vertices are the same colour.

30. Prove that from any set of 10 distinct two-digit numbers, it is possible to select two disjoint subsets whose members have the same sum.

31. Does there exist a convex polyhedron such that no two of its faces have the same number of edges?

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\(^4\)See section 11.2 if you don’t know what this is.
1.1 Logic and deduction

Mathematics is brimming with statements of the form

If $X$, then $Y$.

For example, we could take $X$ to be the statement ‘Rex is a dog’ and $Y$ to be the statement ‘Rex is an animal’. Being such lazy creatures, we mathematicians have invented the following shorthand for such statements, which is often read as ‘$X$ implies $Y$’.

$X \Rightarrow Y$

Now if $X \Rightarrow Y$ and $Y \Rightarrow Z$, then you can automatically deduce that $X \Rightarrow Z$. One of the easiest ways to write a proof is to string together a chain of deductions in this manner, starting with the assumptions of the problem and ending with the conclusion of the problem. This is often called a **direct proof**, an example of which follows.

**Problem** Prove the quadratic formula, which states that if $ax^2 + bx + c = 0$ and $a \neq 0$, then

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

**Solution** Since $a \neq 0$, we can divide both sides by $a$ to obtain

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Now we use an algebraic trick, known as **completing the square**, to write the left-hand side as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right).$$

This is a good time to mention that you should never ever take equations like this for granted. So grab a pen and some paper and check that it’s true for yourself! Once you’ve done that, you should be convinced that the quadratic equation now takes the following form.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Of course, we were careful to consider both the positive and the negative square roots, as one should always do, and this completes the proof.

1.2 Converse

For every statement of the form $X \Rightarrow Y$, there is something known as the **converse**, which is the statement $Y \Rightarrow X$. You may be tempted to think that these two statements mean the same thing, that is, if $X \Rightarrow Y$ is true, then $Y \Rightarrow X$ is true. But this is most definitely not the case. For example, using our two example statements from earlier, it is clear that

Rex is a dog \quad \Rightarrow \quad Rex is an animal
is a true statement, whereas the statement

$$\text{Rex is an animal} \implies \text{Rex is a dog}$$

is false, since there is the possibility that Rex could be a lizard or some other animal. So if we know that a statement is true, there is no guarantee whatsoever that its converse is also true. But sometimes it is, as in the following well-known example.

**Problem** Pythagoras’ theorem states that, if a right-angled triangle has side lengths \(a, b, c\), where \(c\) is the length of the hypotenuse, then \(a^2 + b^2 = c^2\).

Assuming that Pythagoras’ theorem is true, prove the converse of Pythagoras’ theorem.

**Solution** Of course, the first thing we must do is write down what the converse actually is.

If a triangle has side lengths \(a, b, c\), where \(a^2 + b^2 = c^2\), then the triangle is right-angled and \(c\) is the length of the hypotenuse.

Let’s construct a right-angled triangle whose legs have lengths \(a\) and \(b\), and let the hypotenuse have length \(d\). The reason for this is because we can now invoke Pythagoras’ theorem, which we already know to be true. It tells us that \(a^2 + b^2 = d^2\). Using this in conjunction with our assumption that \(a^2 + b^2 = c^2\), we deduce that \(c^2 = d^2\), which implies that \(c = d\).

Therefore, the triangle with side lengths \(a, b, c\) that we were given possesses exactly the same side lengths as the right-angled triangle that we have constructed. This means that the two triangles are, in fact, congruent. So the given triangle was indeed right-angled, as we intended to prove. Furthermore, the equation \(a^2 + b^2 = c^2\) implies that \(c\) is the longest side length in the triangle, and hence is the length of the hypotenuse.

In the previous solution, we relied on Pythagoras’ theorem to prove its converse. This is a rather general strategy, so keep the following point in mind. If you are given a true statement and asked to prove its converse, then it is often advantageous, sometimes crucial, to use the original statement itself.

### 1.3 If and only if

Is there some way to combine Pythagoras’ theorem and its converse into one super-duper Pythagorean statement? Yes, there most certainly is!

**Pythagoras’ theorem and its converse** Suppose that a triangle has side lengths \(a, b, c\), where \(c\) is the longest side. Then the triangle is right-angled if and only if \(a^2 + b^2 = c^2\).

In general, the statements ‘If \(X\), then \(Y\)’ and ‘If \(Y\), then \(X\)’ can be combined to create the single statement ‘\(X\) if and only if \(Y\)’. You can probably guess that the mathematical notation for this is simply

$$X \iff Y.$$ 

Now if someone actually asks you to prove a statement of the form \(X \iff Y\), then what do you do? The simplest approach is to split the problem into two parts. First prove \(X \implies Y\), then prove \(Y \implies X\). The next problem not only demonstrates this point but also provides us with a useful way to test whether or not a number is divisible by 7. You should think about why this is so.
Problem  If $a$ and $b$ are integers, prove that $10a + b$ is divisible by 7 if and only if $a - 2b$ is divisible by 7.

Solution  As with most ‘if and only if’ statements, the proof naturally divides into two parts.

- If $10a + b$ is divisible by 7, then $a - 2b$ is divisible by 7.
  If $10a + b$ is divisible by 7, then certainly $5(10a + b) = 50a + 5b$ is divisible by 7. And if $50a + 5b$ is divisible by 7, then certainly $(50a + 5b) - 7(7a + b) = a - 2b$ is divisible by 7. This proves the statement in one direction.

- If $a - 2b$ is divisible by 7, then $10a + b$ is divisible by 7.
  If $a - 2b$ is divisible by 7, then certainly $(a - 2b) + 7(7a + b) = 50a + 5b$ is divisible by 7. And if $50a + 5b$ is divisible by 7, then certainly $(50a + 5b) \div 5 = 10a + b$ is divisible by 7. This proves the statement in the opposite direction and completes the proof.

Hopefully, you will have noticed that the two parts are very similar in nature. Although this is reasonably common, there will be times when one direction is significantly easier to prove than the other. And, as we mentioned in the previous section, once you’ve proved the statement in one direction, you can often use it to your advantage to prove the statement in the other direction.

1.4 Contrapositive

The contrapositive is a way of turning a logical statement on its head to give an equivalent logical statement. For example, instead of saying

If Rex is a dog, then Rex is an animal,

we could say the equivalent statement

If Rex is not an animal, then Rex is not a dog.

In general, the contrapositive of the statement $X \Rightarrow Y$ is the equivalent statement

‘not $Y$’ $\Rightarrow$ ‘not $X$’,

where ‘not $X$’ is the opposite of $X$ and ‘not $Y$’ is the opposite of $Y$. By calling a statement and its contrapositive equivalent, we mean that if the statement is true, then its contrapositive is true, while if the statement is false, then its contrapositive is false. In other words, proving either one will automatically prove the other. Note that if you take the contrapositive of the contrapositive, then you actually end up with the statement you started with. As an example, consider the following two statements, which are the contrapositives of each other.

- If a shape is a rectangle, then it has four sides.
- If a shape does not have four sides, then it is not a rectangle.

Problem  If $a$ and $b$ are real numbers such that $ab$ is irrational, then at least one of $a$ and $b$ must be irrational.